ASSIGNMENT 1

FORMAL PROOFS AND THE LIMITS OF COMPUTATION
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Chomsky Normal Form

Exercise 1 (Chomsky Normal Form). *Let G be a context-free grammar. Give an algorithm, taking $\langle G \rangle$ as input which outputs a grammar G' in Chomsky normal form such that $L(G) = L(G')$.*

Exercise 2 (Bounded Derivations). *Consider G, a context free grammar in Chomsky Normal Form, and w, such that $|w| = n$. Show that any derivation of w necessarily has length less than or equal to $2n - 1$.*

Finite Automata can do addition

Exercise 3 (DFA for Addition). *Consider the alphabet*

$$\Sigma = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

consisting of all 8 3-bits vectors. A string of symbols in Σ can be understood as specifying 3 integers a_1, a_2, a_3 in base 2, each corresponding to reading off a different row.
Construct a DFA M such that $L(M) = \{w \in \Sigma^* \mid a_1 + a_2 = a_3\}$

Building the Smallest, Biggest Dam

Exercise 4 (The $(2,2)$ Busy beaver). *Build a Turing machine with 2 states and tape alphabet $\{1\} \cup \{B\}$ which, on an input consisting of all blank symbols B, halts with the longest string of 1s on its tape.*

Exercise 5 (The $(3,2)$ Busy beaver). ***Challenge:*** *Build a Turing machine with 3 states and tape alphabet $\{1\} \cup \{B\}$ which, on an input consisting of all blank symbols B, halts with a string consisting of as many 1s as possible on its tape.*

Rice Theorem

Exercise 6 (Any Non-Trivial Question about Turing Machines is Undecidable). *A language property $\mathcal{R}$ is called non-trivial if there exist two Turing-recognizable languages that respectively do and do not satisfy $\mathcal{R}$. Show that, for any non-trivial property $\mathcal{R}$, the set of all Turing machines whose language verifies property $\mathcal{R}$ is undecidable, namely:*

$$L_{\mathcal{R}} = \{\langle M\rangle \mid \mathcal{L}(M) \in \mathcal{R}\}$$

is undecidable.

Remark. Use this to show that the language:

$$\{\langle G|G'\rangle \mid G, G' \text{ context free grammars}, L(G) = L(G')\}$$

is undecidable.

Closure Properties of Regular Languages/Finite Automata

Exercise 7 (Union). *Recall that a language L is called regular if there exists a deterministic finite automaton M such that $L = L(M)$. Show that for any two regular languages A and B, $A \cup B$ is also regular.*

Exercise 8 (intersection). *Show that for any two regular languages A and B, $A \cap B$ is also regular.*

Exercise 9 (Complement). *Show that if L is a regular language, then so is L^C.*

Exercise 10 (Non-Determinism does not grant additional computation power). *Consider a non-deterministic finite automaton N. Construct a deterministic finite automaton D such that $L(D) = L(N)$.*