NON-OBTUSE DISSECTIONS OF TETRAHEDRA

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Abstract. We prove that every tetrahedron can be dissected into non-obtuse tetrahedra.

1. Introduction

In the plane any obtuse triangle can be dissected into 2 right-angled triangles by drawing the altitude from the obtuse vertex on its opposite side. The problem generalises naturally to higher dimensions. We consider a simplex to be *non-obtuse* provided no angle between two co-dimension 1 faces is obtuse. The problem at hand is then to find a *non-obtuse dissection* of a simplex, i.e a partition into non-obtuse sub-simplices. Unlike in the two-dimensional case, there seems to be no equivalent trivial scheme to obtain non-obtuse dissections of n-simplices ($n > 2$). For example, consider the simplex with vertices $(-1,-1,+\epsilon)$, $(-1,1,-\epsilon)$, $(1,-1,-\epsilon)$ and $(1,1,+\epsilon)$ and notice that for ϵ small enough none of the vertices have orthogonal projections lying on their opposite face. In three dimensions in particular, dihedral angles correspond to angles in the spherical triangles formed by the link of the tetrahedron's vertices. One is then lead to dissect obtuse spherical triangles into non-obtuse ones. As evidenced by Itoh and Zamfirescu's work on minimal acute triangulations of spherical triangles [JaT02], this is no longer a trivial problem. Our main theorem thus offers a constructive answer to this interesting puzzle:

Theorem 1.1 (Main Theorem). *There exists a dissection of any tetrahedron into (at most 28) non-obtuse tetrahedra.*

Organisation. Every planar triangle can be decomposed in 6 right-angled triangles meeting at the centre of its inscribed circle (Figure 1). Our preliminary dissection is a natural extension to 3 dimensions of this simple decomposition (Section 3). Unlike in the plane however, this first dissection needs to be further refined to guarantee non-obtuseness of its tetrahedra (Section 4). First, we establish some elementary results to help characterize non-obtuse tetrahedra (Section 2).

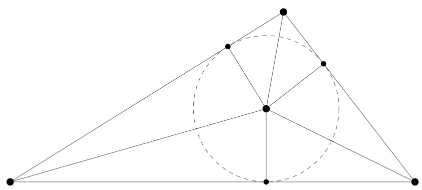

Figure 1

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2. Non-Acute Cycles

In this section and throughout the rest of the article, a, b, c and d denote the four vertices of a tetrahedron. For $x, y \in \{a, b, c, d\}$, we say that the edge xy is *non-acute* if the dihedral angle at xy is non-acute. In this section only, we name l the line ac and p the orthogonal projection of d onto the plane abc. We begin by establishing two lemmas which will allow us to later verify that the dissection introduced is non-obtuse. Although non-essential to our proof, we also draw as corollary a simple classification of tetrahedra into 6 distinct types.

Lemma 2.1. *If the dihedral angles at ab and bc are non-acute, then p lies in the intersection of the two half-planes delimited by the lines ab and bc which do not contain the triangle abc.*

Proof. Fix and name $\vec{n}$ one of the two unit normal vectors to the plane abc. Since the vectors $\vec{ab}, \vec{cb}, \vec{n}$ form a basis of $\mathbb{R}^3$, we can write $\vec{bd}$ as a linear combination $\alpha\vec{ab} + \beta\vec{cb} + \gamma\vec{n}$, where $\alpha, \beta, \gamma \in \mathbb{R}$. The conditions on the dihedral angles at ab and bc then become respectively $\beta \geq 0$ and $\alpha \geq 0$, which proves the lemma. $\square$

Using the same notation as in Lemma 2.1 we prove the following:

Lemma 2.2. *If the dihedral angles at ab and bc are non-acute, then $dist(l, p) \geq dist(l, b)$.*

Proof. Name $\vec{m}$ the unit normal vector to the line ac contained in the plane abc and pointing towards the interior of the triangle. Observe then that $\langle \vec{m}, \vec{bd} \rangle = \langle \vec{m}, \alpha\vec{ab} + \beta\vec{cb} + \gamma\vec{n} \rangle = \alpha\langle \vec{m}, \vec{ab} \rangle + \beta\langle \vec{m}, \vec{cb} \rangle \geq 0$, since $\langle \vec{m}, \vec{ab} \rangle > 0$ and $\langle \vec{m}, \vec{cb} \rangle > 0$. This proves our claim that $dist(l, p) \geq dist(l, b)$. $\square$

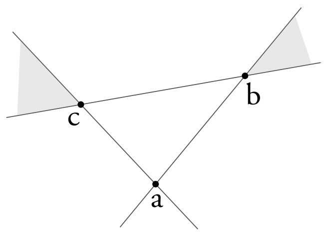

Figure 2

Lemma 2.3. *There are no non-acute 3-cycles in the 1-skeleton of a tetrahedron.*

Proof. Without loss of generality, consider the triangle abc and suppose by contradiction that its boundary forms a 3-cycle of non-acute edges. Applying Lemma 2.1 to the pairs of edges ab, bc and bc, ca we see that p is forced to lie in the intersection of 2 disjoint regions of the plane (Figure 2), which is a contradiction. $\square$

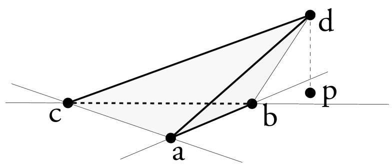

Figure 3

Lemma 2.4. *There are no non-acute 4-cycles in the 1-skeleton of a tetrahedron.*

Proof. Suppose by contradiction that the edges ab, bc, cd and da form a non-acute 4-cycle (indicated in bold in Figure 3). Denote by l the line ac and by p and q the points obtained by projecting d on the plane abc and b on the plane acd. Because the angles ab and bc are non-acute, Lemma 2.2 yields the inequality $dist(b,l) \leq dist(p,l)$. Furthermore $dist(p,l) \leq dist(d,l)$ as p is the orthogonal projection of d. Symmetrically, since cd and da are also non-acute, we obtain the inequality: $dist(d,l) \leq dist(q,l) \leq dist(b,l)$. This implies that $d = p$, which is impossible. □

Corollary 2.5. *The subgraph formed by the non-acute edges of a tetrahedron is isomorphic to one of the following 6 graphs (the underlying K_4 graph formed by the 1-skeleton of the tetrahedron is drawn in thin lines):*

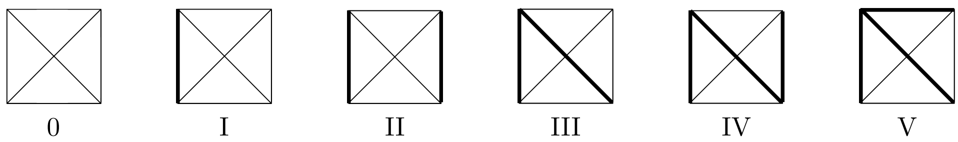

FIGURE 4

Proof. If there are no non-acute edges the subgraph of non-acute edges is empty (0). Because of Lemmas 2.3 and 2.4, the subgraph formed by non-acute edges contains at most 3 edges and must be a forest. Up to an automorpishm of K_4, the only possible subgraph with one non-acute edge is (I). If two edges are non-acute then either the edges are opposite (II) or they are adjacent (III). Finally, a forest with 3 edges on 4 vertices is a tree and there are only two trees with 3 edges up to automorphism, namely a path of length 3 (IV) and a claw (V). □

3. THE INSCRIBED SPHERE DISSECTION

We now introduce a first preliminary dissection of a tetrahedron into at most 24 sub-tetrahedra. To begin, consider the centre o of the inscribed sphere to $abcd$ (see [Ber09, 10.6.8] for a proof of its existence and uniqueness). Throughout the rest of the article we name p and q its orthogonal projections onto the planes (abc) and (abd) (they always exist since the inscribed sphere is tangent to each face of $abcd$), and r the projection of o onto the open edge ab (if it exists).

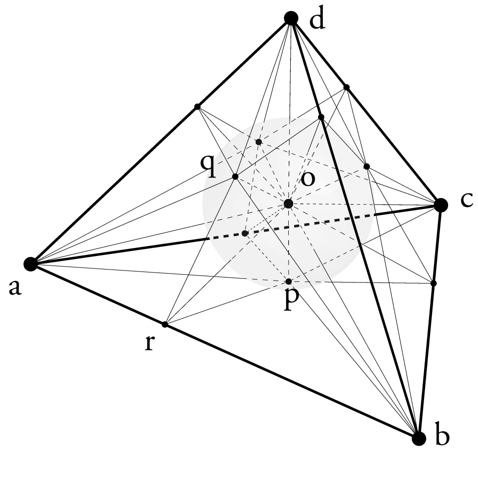

FIGURE 5

This first dissection is determined entirely by points on the boundary of $abcd$. Namely, all the sub-tetrahedra are constructed by taking the cone of vertex o and base the triangle formed by the projection of o on one of the faces, its projection on one of the edges of the selected face (if it exists), and one of the endpoints of the selected edge. Assuming all edge projections exist and going through all possible triples of (face, edge, vertex), this construction yields a decomposition into 24 tetrahedra (4 for each edge, or 6 for each face).

Lemma 3.1. *If the projection r of o onto the edge ab exists, then the tetrahedron $arpo$ is non-obtuse.*

Proof. By construction, the dihedral angles at ap, pr and ro are all right angles. Since there are no non-acute 3-cycles both op and ar are acute. Similarly, since there are no 4-cycles, it must be that ao is also non-obtuse. □

Lemma 3.1 shows that if o projects onto all edges, this preliminary dissection is already non-obtuse.

Unfortunately, the projection of o onto every edge of $abcd$ is not guaranteed. To see why, remember from linear algebra that projecting o onto the edge say ab or projecting p onto that same edge yield the same point. Thus if $abcd$ is such that the face abc (for example) forms an obtuse triangle, p — and therefore o — does not necessarily projects onto one of the sides of abc. It is possible then that either the projection of p onto ab is degenerate and coincides with either a or b, or it simply does not exist. In the former case, Lemma 3.1 guarantees that the tetrahedron $abpo$ is non-obtuse, and similarly for the opposite (identical) tetrahedron $abqo$.

If, however, o does not project onto a given edge, we replace the 4 corresponding sub-tetrahedra in our preliminary dissection by the 2 tetrahedra formed by taking the 2 cones of vertex o and base the triangle with vertices the projection of o on either one of the two incident faces to that given edge and the two endpoints of that same edge. Unfortunately these new tetrahedra are not guaranteed to be non-obtuse and might need to be further dissected. Fortunately however, the following claim guarantees that at most 4 such tetrahedra might require our attention:

Claim 3.2. *There are at most 2 edges on which o does not project, furthermore if such is the case, then they must be opposite edges.*

Proof. Select an edge, say ab, and suppose that o does not project onto it. Observe that any interior point of a euclidean triangle always projects on at least two sides since for any given point of the triangle, the two regions corresponding to points without projections on either one of the two edges meeting at that point are disjoint. So it must be that o has projections on both bc, ac, and bd, da and there remains only the edge cd onto which (possibly) o does not project. □

4. Proof of the Main Theorem

Starting with the inscribed sphere dissection prescribed in section 3, Lemma 3.1 guarantees that this preliminary dissection is non-obtuse as long as o projects onto all the edges of $abcd$ (even if that projection is degenerate). Suppose then without loss of generality that o does not project onto the edge ab - if o is also missing a projection onto the opposite edge cd, the corresponding tetrahedra will be dealt with accordingly with the same procedure. The task at hand is then to dissect the tetrahedra $apbo$ and $abqo$. Since both tetrahedra are identical, we restrict our attention to $abpo$ and proceed to prove Proposition 4.1, from which our main theorem follows directly.

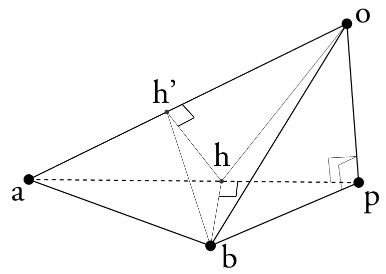

FIGURE 6

Proposition 4.1. *There exists a dissection of abpo into 3 non-obtuse tetrahedra.*

Proof. We know that either the angle $\angle abp$ or the angle $\angle pab$ is obtuse as o does not project onto ab. Suppose without loss of generality that $\angle abp$ is obtuse (Figure 6). Observe first that the dihedral angle at ap is a right angle and thus the orthogonal projection h of b onto the plane apo lies on the line ap. Furthermore our assumption that $\angle abp$ is obtuse guarantees that h lies in the open edge ab. Observe both projections of b onto the edges ao (h' in Figure 6) and op exist: p is the orthogonal projection of o onto abc and thus both $\angle bpo$ and $\angle apo$ a right angle. This also tells us that the projection onto op coincides with p. Lemma 3.1 then guarantees that this dissection is non-obtuse (replacing o with b) and since the triangle apo has been partitioned in 3 sub-triangles this construction yields 3 sub-tetrahedra. ☐

As noted in Claim 3.2, at most 2 edges can be missing projections from the center of the inscribed sphere, in which case there is a need to further dissect 4 sub-tetrahedra using Proposition 4.1. In total this yields a dissection of any tetrahedron $abcd$ into at most $24 - 4 \times 2 + 4 \times 3 = 28$ sub-tetrahedra.

5. ACKOWLEDGMENTS

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REFERENCES

[JaT02] J.I.Itoh and T.Zamfirescu, *Acute triangulations of triangles on the sphere*, Rendicounti Del Circolo Matematico Di Palermo Serie II (suppl. 70) (2002), 59–64.

[YaV60] Y.Burago and V.Zalgaller, *Polyhedral embedding of a net*, Vestnik Leningrad. Univ **15** (1960), no. 7, 66–80.

[Gar95] Martin Gardner, *New mathematical diversions*, Mathematical Association of America, 1995.

[Man60] Wallace Manheimer, *Solution to problem E1406: Dissecting an obtuse triangle into acute triangles*, Amer. Math. Monthly **67** (1960), 923.

[Zam02] Tudor Zamfirescu, *Acute triangulations: a short survey*, Proceedings of the VI Annual Conference of the Romanian Society of Mathematical Sciences, 2002, pp. 10–18.

[EaIaP12] E.Kopczyński and I.Pak and P.Przytycki, *Acute triangulations of polyhedra and* $\mathbb{R}^n$, Combinatorica **32** (2012), no. 1, 85–110.

[Sar09] Shubhangi Saraf, *Acute and nonobtuse triangulations of polyhedral surfaces*, European Journal of Combinatorics **30** (2009), no. 4, 833–840.

[Ber09] Marcel Berger, *Geometry*, Universitext, vol. 1 and 2, *4th revised edition*, Springer-Verlag, Berlin Heidelberg, 2009.

[Cox73] H.S.M Coxeter, *Regular polytopes*, Courier Corporation, 1973.

[Pak08] Igor Pak, *Lectures on discrete and polyhedral geometry*, 2008.

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