MY KINGDOM FOR A CYCLOGON!
THE QUADRATURE OF A DISCRETE CYCLOID

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ABSTRACT. We give an elegant formula for the area under the curve traced by a chosen vertex of a regular n-gon, rolling without friction on a line. Namely, we show that it is equal to the area of the polygon itself plus twice that of its circumscribed circle. Taking the limit as n goes to infinity provides a new proof of the formula for the quadrature of the cycloid. The proof involves a generalisation of the Pythagorean theorem which might be noteworthy on its own.

Remark. Since posting this short proof, I was made aware that this is a rediscovery of [1] and adopted their terminology in the title to reflect it.

Consider a regular n-gon $\mathcal{P}_n$ inscribed in the unit circle $\mathcal{C}$. As $\mathcal{P}_n$ rolls without friction on a line l, a fixed vertex V of $\mathcal{P}_n$ traces a piecewise continuous curve γ_n (called a *cyclogon* in [1]) between any two successive points of contact of V with l (see Figure 1).

Theorem. *The area A_n of the region delimited by l and γ_n is equal to twice that of the circumscribed circle $\mathcal{C}$ to $\mathcal{P}_n$ plus the area of the polygon itself:*

$$A_n = \text{Area}(\mathcal{P}_n) + 2\,\text{Area}(\mathcal{C})$$

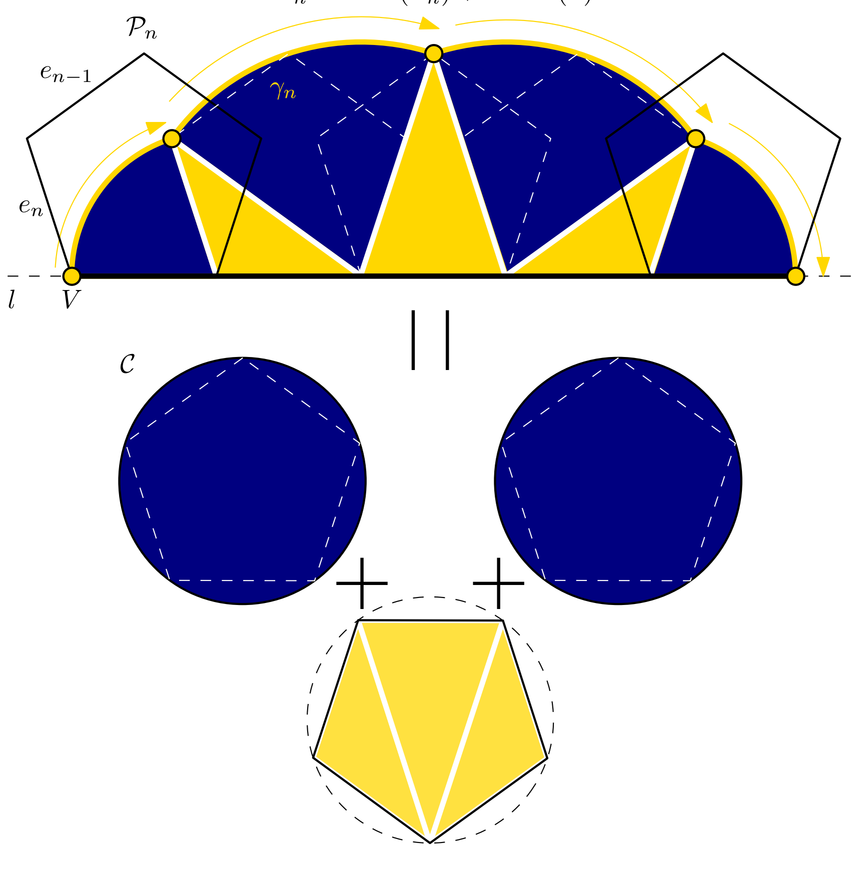

FIGURE 1: The crown's area (in gold) equals that of the polygon itself, while the blue area totals twice that of its circumscribed circle.

Proof. Order the edges of $\mathcal{P}_n$ counter clockwise from e_1 to e_n and assume $\mathcal{P}_n$ rolls on l without friction towards the right (see Figure 1). The movement of $\mathcal{P}_n$ is then a succession of n identical steps, where step i starts off with e_i lying flat on l, followed by $\mathcal{P}_n$'s rotation at an angle of $\frac{2\pi}{n}$ about the rightmost endpoint of e_i, until e_{i+1} lies flat on l. Imagine then a fixed, immobile regular n-gon $\mathcal{P}_n^{\text{ref}}$ which we translate and superimpose on $\mathcal{P}_n$ at the beginning of each step i. Label its vertices V_1 to V_n (see Figure 2). Effectively, each step i sees V move from V_i to V_{i+1}. The area A_n can thus be decomposed into two parts: a golden *crown* consisting of $n-2$ triangles, each joining the base of $\mathcal{P}_n^{\text{ref}}$ with one of the $n-2$ vertices which do not lie on the base; and a collection of $n-1$ blue angular sectors, each of angle $\frac{2\pi}{n}$ and radius equal to the *diagonal* d_i joining V_n to V_i, $i \in [n-1]$. It is clear that the area of the golden crown is that of the regular n-gon itself. We are left to compute the blue area. This area is equal to $\sum_{i=1}^{n-1} \frac{\pi}{n} d_i^2$.

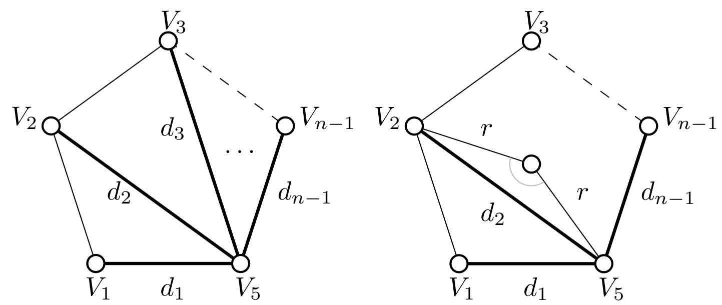

FIGURE 2: The numbering scheme for the polygon $\mathcal{P}_n^{\text{ref}}$ (left) and the isosceles triangle used for the cosine rule (right).

Using the cosine rule in the isosceles triangle with base of length of d_i and twin sides of unit length (see Figure 2), we get that $d_i^2 = 2(1 - \cos\frac{2\pi}{n})$. Recalling the fact that the sum of the n-th roots of unity totals 0 – or equivalently that the sum of the n-th roots of unity excluding 1 is -1 – this directly leads to:

$$\frac{1}{n}\sum_{i=1}^{n-1} d_i^2 = \frac{1}{n}\sum_{i=1}^{n-1} 2\left(1 - \cos\frac{2\pi}{n}\right) = 2\frac{(n-1)-(-1)}{n} = 2$$

Multiplying both sides of the previous equation by π yields the proof of the theorem.

□

Corollary. *The quadrature of the cycloid is equal to thrice the area of the rolling circle.*

We end with a remark that our previous computation shows a fact that is perhaps interesting on its own, namely that the radius of a circle averages the diagonals of an inscribed polygon in the following sense:

Theorem (Generalised Pythagorean Theorem). *Let $\mathcal{P}$ be a regular polygon inscribed in a circle $\mathcal{C}$. The square of the radius of $\mathcal{C}$ is equal to half of the average of the squared distances from one vertex of $\mathcal{P}$ to all the other vertices of $\mathcal{P}$.*

Remark. When the polygon is a square, this is equivalent to the Pythagorean theorem.

References

[1] T.M. Apostol and M.A. Mnatsakanian. Cycloidal areas without calculus. *Math Horizons*, 7(1):12–18, 1999. ↑1

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