COUNTING TANGENT TRIANGLES INSCRIBED IN A PLANE CURVE

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Abstract. Fabricius-Bjerre's formula gives a combinatorial formula relating the number of different types of bitangents of a plane curve to other simple invariants of that curve (namely, its number of inflection points and self-crossings). A natural generalisation might be to consider tangent polygons inscribed in a curve instead of bitangents (which we can think of as 2-gons). In the simplest extension, we would like to obtain a similar combinatorial formula for tangent triangles inscribed in a curve. This requires understanding what the different kinds of tangent triangles should be, as well as what invariants they ought to be related to.

Consider a sufficiently regular plane curve Γ (how regular we really need it to be will be determined later on). For now, we will consider Γ to be the geometric image of a parametrized curve γ and assume γ to be C^∞. Likewise, we shall assume all inflection points, self-crossings and bitangents are regular enough for the present discussion to make sense (e.g. non-zero curvature at bitangents and the likes..).

1. Two, Four, and Eight Types of Triangles

In the original proof of Fabricius-Bjerre, 6 types of bitangents are distinguished, depending on which side the curve is on at the 2 points of tangency, and which orientation the curve possesses at each point. This should result in $4^2 = 16$ choices, but only 8 if we do not wish to distinguish between the two vertices, and only 6 up to planar direct isometry.

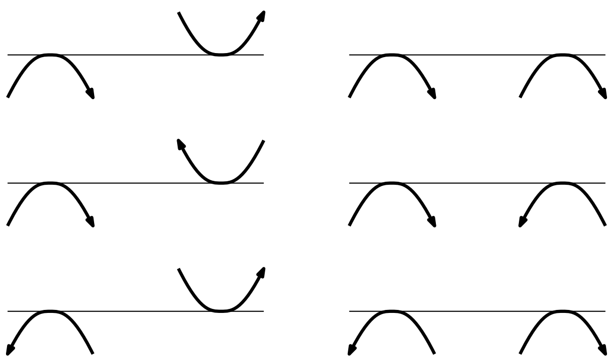

Figure 1: The 3 different types of positive/negative bitangents on the left/right.

It should be noted that while the 6 different types play a role in the proof, where two identities obtained for direct and indirect tangent rays give two relations between the various types, in the end the identity ignores the orientation of the curves at the points of tangency and only cares about the geometric information of whether the curve is on either side of the bitangent or not.

Following this approach, it seems sensible to distinguish 4 types of tangent triangles (where we previously had 2 types of bitangents).

Definition 1.1 (4 Triangle Types)**.** Given a tangent triangle T inscribed in the plane curve Γ, we begin by giving each side of T an orientation, according to the direction of the tangent ray that supports it. We then assign to each side the colour blue (resp. red) depending on whether the curve lies in the positive (resp. negative) half-plane supported by the associated oriented tangent ray. There are 2^3 distinct 2-colourings of a triangle, but only 4 are distinct up to direct planar isometries. These are the 4 types of triangles we shall consider.

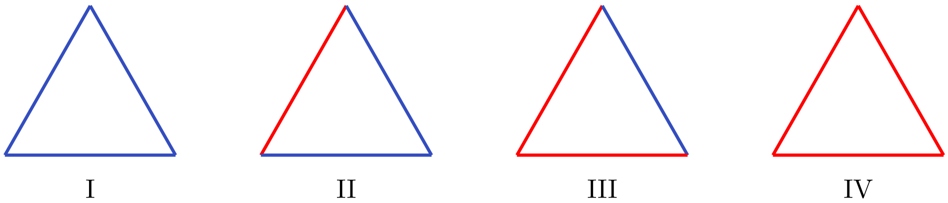

FIGURE 2: The 4 types of tangent triangles.

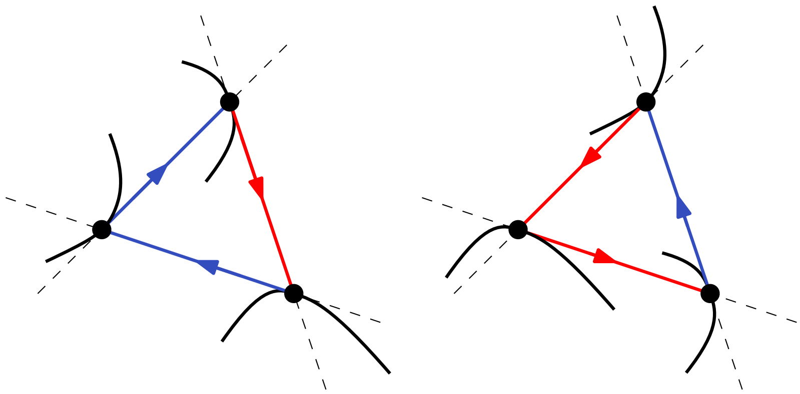

FIGURE 3: Example of a tangent triangle of type II_b (left) and III_b (right).

Definition 1.2 (8 Triangle Types)**.** Each family of triangles can be further subdivided into 2 sub-families, denoted by subscripts a or b. Depending on the parity of the number of triangle vertices at which the curve lies inside/outside the tangent triangle, where a positive sign is assigned if the curve lies inside, we assign the subscript a (resp. b) to an even (resp. odd) parity.

Observation The previous parity is in fact the same as the orientation of the tangent triangle, since there can only be 2 solutions. Thus for example, Type II_a corresponds to a counter-clockwise oriented tangent triangle, while II_b corresponds to a clockwise oriented one.

This parity affects the sign of the derivative of τ^3, where τ is the multifunction which maps the parameter t of a point $\gamma(t)$ to the parameter(s) of the intersection point(s) between the tangent to Γ at $\gamma(t)$ and Γ itself. (Note that the tangent triangles correspond to fix points of τ^3). For example, a type II_b or II_b would correspond to a negative derivative (see for example Fig. 4) and thus a locally unique fixed point for τ^3, while type II_a and II_a correspond to a positive derivative and thus might lead to complications (see Fig 3 and Fact II in section 2).

Remark. It is also worth noting - as added justification to make this definition less arbitrary - that the determinants used to define the triangle types pop up in Halpern's proof of the Fabricius-Bjerre formula if we were to adjust the vector field whose 0 are counted in a natural way.

Yet, after running computer experiments, it seems what matters most is the following definition:

Definition 1.3 (2 Triangle Types)**.** A tangent triangle is *positive* (resp. *negative*) depending on the even (resp. odd) parity of its number of blue sides (i.e., triangles of type II and IV are positive, and I and III negative).

Computer experiments seemed to very strongly suggest the following conjecture:

Conjecture 1.4 (False conjecture, but somehow mostly true)**.** *The number of positive tangent triangle is always equal to the number of negative tangent triangles.*

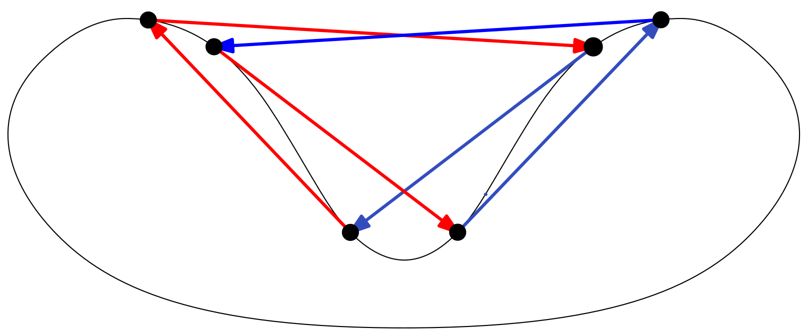

FIGURE 4: The conjecture is easily verified for all curves with at most two inflection points and no self-crossings.

2. Counter Examples

The natural way to go about this problem when doing computer experiments is to discretize the setting and look at polygonal curves, especially since Banchoff gave a simple elementary proof of the Fabricius-Bjerre's formula using deformation arguments for polygonal curves. Doing so, it is much easier to generate and dream up counter examples, which in this section will be stated for polygonal curves.

Fact 0. *There exists a polygonal curve for which an elementary deformation changes the count of positive tangent triangles by 1, while leaving the count of negative tangent triangles, self-crossings and inflection points unchanged.*

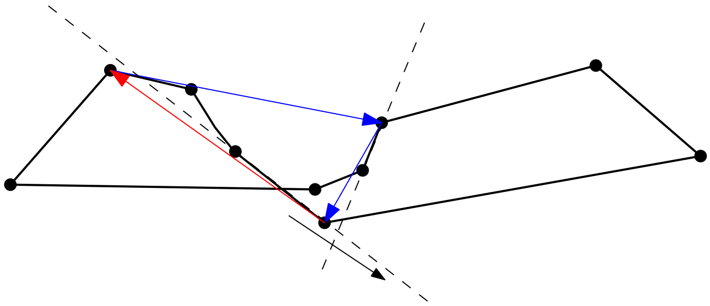

FIGURE 5: The underpinned elementary move eliminates the depicted type II triangle and does not create any other triangle or self-crossing or inflection point.

Fact I. *There exists a polygonal curve with only two tangent triangles, one triangle of type* I_a *and one triangle of type* II_b.

(Insert image)

Fact II. *If a curve admits a triangle of type a (of any of the 4 types), we can "blow up" this triangle into arbitrarily many triangles of the same type, locally.*

The justification of Fact II is given by figure 5, where we can devise a recursive subdivision/smoothing scheme to generate arbitrarily many "duplicates" of an original pair of tangent triangles. In the continuous setting, this corresponds to the fact that in the situation where a type II_a triangle exists, nothing prevents the second derivative of the function τ^3 from being positive. In practice, it is possible to adjust the curvature and position of Γ so that τ^3 is the identity of an interval. This can be obtained for example by considering the limit of the recursive subdivision scheme, since the associated limiting curve is smooth and has all its vertices part of a tangent triangle.

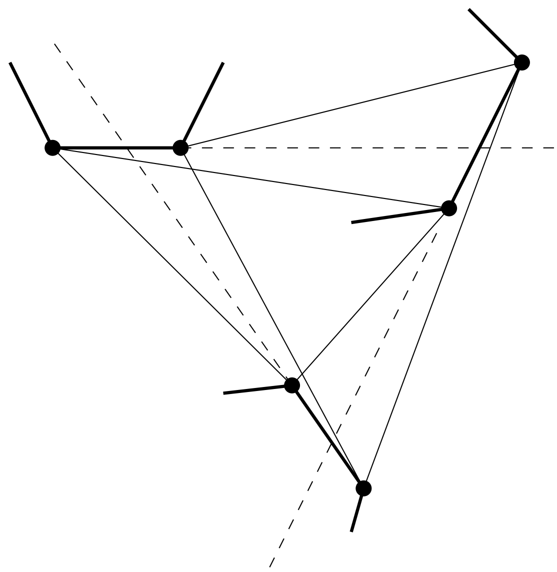

FIGURE 6: Example of a doubling of a type II_a triangle in the discrete setting.

Together with Fact I, Fact II yields the following fact without too much effort:

Fact III. *There exist polygonal curves which have an arbitrarily large gap between their number of positive and negative tangent triangles.*

Remark. Fact III might not be so interesting, but more akin to the regularity requirement that bitangents cannot have 0 curvature in the Fabricius-Bjerre setting. In some sense, Fact 0 is perhaps more disheartening.

3. Curve Types and New Conjectures

Thinking more about Fact 0 brought the following objects and questions to mind.

Definition 3.1 (Curve Type)**.** To every curve Γ, we associate a multigraph G_Γ with vertex set its set of inflection points and self-crossings, where we insert a distinct edge between two vertices u and v (or a loop based at a single vertex) for each distinct path in Γ joining u and v that does not contain an inflection point or a self-crossing. We then 2-colour the vertices of G_Γ depending on whether they are inflection points or self-crossings. The colour-preserving isomorphism class of G_Γ is called the *curve type* of Γ.

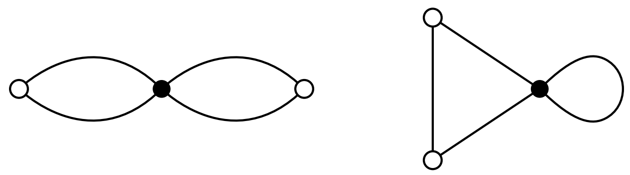

Figure 7: The only two curve types with 1 self-crossing and two inflection points.

For any given curve type, there is a representative curve Γ which minimises the number of positive tangent triangles and one which minimises the number of negative tangent triangles.

Conjecture 3.2. *Every curve type has a representative that minimises the number of positive and negative triangles simultaneously. Moreover, for such a representative, the number of positive tangent triangles is equal to the number of negative tangent triangles.*

Vague Questions Since our first conjecture fails, why is it at all the case that computer experiments return at most a gap of 1 or 2 between the positive and negative triangle counts? (Even on complicated curves with more than a hundred of tangent triangles). Could we show that the identity is only false for a set of curves of measure 0? What is a random polygonal curve? How do we put a measure on the space of such polygonal curves? Maybe easier with linkages since their moduli spaces are finite dimensional?

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