TANGENT POLYGONS INSCRIBED IN A PLANE CURVE

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abstract. I investigate the tangent polygons inscribed in a plane curve (see Fig. 1 for an example of a 3-gon). In particular I wonder if such polygons are common and if one can devise a numbering scheme based on the geometrical properties of the curve (when they are countable), or perhaps in the form of a combinatorial formula analogue to the Fabricius-Bjerre-Halpern formula (or its integral version). Focusing on a specific configuration I am able to reduce the problem to the study of a one-dimensional dynamical map and glimpse into the fractal structure of the set of points formed by the vertices of such polygons. In particular I show that two inflection points creating a non-convex well shape in the curve are enough to give this set a fractal structure - that of a Cantor set.

Frame

The frame for this discussion and the conditions of regularity for the curves we consider are essentially the same ones as that presented in Halpern's formulation of the Fabricius-Bjerre formula (see section 1.1 of [1]). We simply add what seems like a natural definition for tangent polygons and the regularity conditions that go with it.

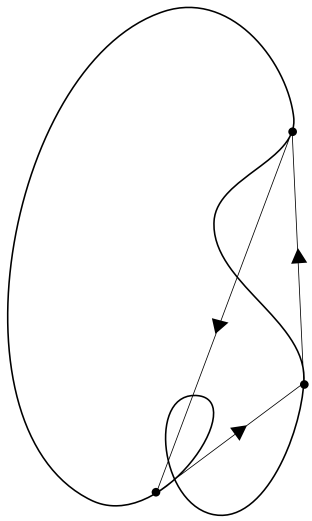

Figure 1: Tangent 3-gon

Let then $\mathbb{S}^1$ denote the one dimensional sphere obtained as $\mathbb{R}/L\mathbb{Z}$, $L \in \mathbb{R}_+^*$. We consider the smooth, closed plane curve defined as the smooth immersion $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ and denote by $\Gamma := \gamma[\mathbb{S}^1]$ its locus.

Let $T = \frac{\gamma'}{\|\gamma'\|}$, $N = JT$ and $\kappa = \langle N, T' \rangle$ be the unit tangent vector, the principal normal vector and the signed curvature of γ, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let square brackets denote the determinant, i.e for all $u, v \in \mathbb{R}^2$:

$$[u, v] := Det(u, v) = \langle u, Jv \rangle$$

I now introduce the same terminology introduced by J.Barbara in her thesis on a new proof of Fabricius-Bjerre formula ([1]), in which my definition for tangent n-gons find a natural fit.

(Definitions/Frame). (i) An unordered pair $\{s,t\} \subset \mathbb{S}^1$ with $s \neq t$ is called a double point or a crossing if $\gamma(s) = \gamma(t)$. A double point is called regular if the intersection is transverse, i.e if $[\gamma'(s), \gamma'(t)] \neq 0$. Denote by $cr = cr(\gamma)$ the number of regular double points of γ.

(ii) A point $\{s\}$ is called an inflection point of γ if $\kappa(s) = 0$. An inflection point is called regular if $\kappa'(s) \neq 0$. The number of regular inflection points is denoted by $\text{infl} = \text{infl}(\gamma)$.

(iii) An unordered pair $\{s,t\} \in \mathbb{S}^1$ with $s \neq t$ is called a bitangent pair if it is not a double point and the tangent lines at $\gamma(s)$ and $\gamma(t)$ coincide, i.e. if :

$$\gamma(t) \neq \gamma(s) \text{ and } \begin{cases} [\gamma'(s), \gamma'(t)] & = 0 \\ [\gamma'(s), \gamma(s) - \gamma(t)] & = 0 \end{cases}$$

A bitangent pair is called regular if both points have non zero curvature, i.e $\kappa(s)\kappa(t) \neq 0$.

(iv) A set of n distinct parameters $\{t_1, t_2, \dots, t_n\} = \Omega \subset \mathbb{S}^1$ is called a tangent n-gon if :

- For all $a, b \in \Omega$, $\{a, b\}$ is not a double point
- There exists a sequence $\omega_1, \omega_2, \dots, \omega_n$ of n distinct elements of Ω such that $\forall i \in [\![1, n]\!]$, $[\gamma'(\omega_i), \gamma(\omega_i) - \gamma(\omega_{i+1[n]})] = 0$

A tangent n-gon is called regular if $\kappa(\omega_1)\kappa(\omega_2)\dots\kappa(\omega_n) \neq 0$.

We want to keep our notion of n-gon as general as possible so we won't impose the following condition : $\forall i, j \in [\![1, n]\!], j \neq i + 1[n] \Rightarrow [\gamma'(\omega_i), \gamma(\omega_i) - \gamma(\omega_j)] \neq 0$. Thus we allow our n-gons to look somewhat funky.

As expected, when $n = 2$, 2-gons and bitangent pairs are one and the same.

From now on I will only concern ourselves with generic curves, in the following sense :

(Generic curve). Let $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$. We say that γ is a generic curve if all the double points, bitangent pairs, inflection points and n-gons are regular.

In that context we have the following remarkable combinatorial formula :

(Fabricius-Bjerre's Theorem). Let $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ be a generic curve (in this instance there is of course no need for the n-gons to be regular). Then cr, ext, int and infl are finite and the following relation holds :

$$ext = int + cr + \frac{1}{2}\text{infl}$$

Tangent Rays

The few results I will present here stem for my study of the dynamical system which corresponds to iterating tangent rays on the curve, i.e I wish to study the map that links a point on the curve to the intersection point of its tangent to the curve with the curve itself (fig. 3). In this section I try to define this properly.

The cartesian equation of the tangent to the curve γ at the point of parameter t is :

$$\left[\gamma'(t), \gamma(t) - \begin{pmatrix} x \\ y \end{pmatrix}\right] = 0$$

For further clarity I define T_t :

$$T_t : \begin{cases} \mathbb{R}^2 & \longrightarrow \mathbb{R} \\ u & \longmapsto \left[\gamma'(t), \gamma(t) - u\right] \end{cases}$$

Then the tangent to the curve γ at the point of parameter t can be simply written as :

$$\left\{u \in \mathbb{R}^2 | T_t(u) = 0\right\}$$

Since all the bitangent pairs are regular we know that for a fixed t, the set of the intersection points of the tangent at t to γ with γ itself is finite : $|\left\{u \in \mathbb{R}^2 | T_t(u) = 0 \wedge u \in \Gamma\right\}| = k$ for some $k \in \mathbb{N}$.

Suppose γ has p bitangents and partition $\mathbb{S}^1$ into the corresponding $2p$ intervals $I_1, \ldots, I_{2p}$:

$$\left\{u \in \mathbb{R}^2 | T_t(u) = 0 \wedge u \in \Gamma\right\} = \bigcup_{i \in [\![1,2p]\!]} \left\{u \in \mathbb{R}^2 | T_t(u) = 0 \wedge u \in \gamma\left[I_i\right]\right\}$$

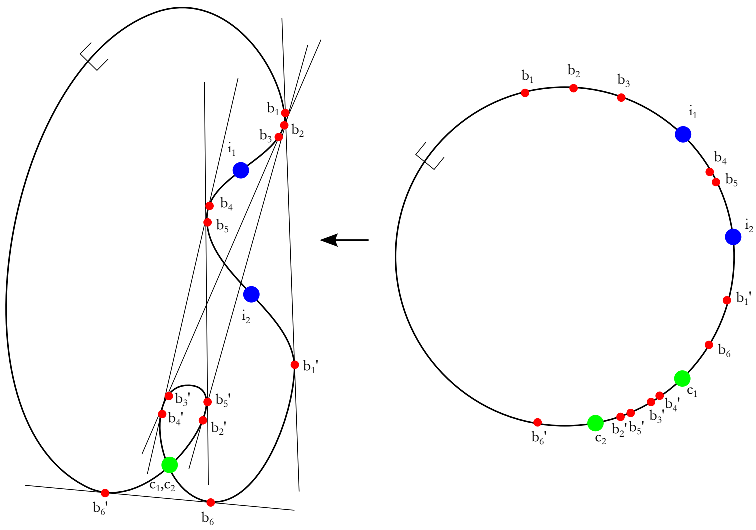

Figure 2: Partition of $\mathbb{S}^1$ according to the bitangents of γ, double points in green and inflexion points in blue

Now on each of these $2p$ intervals, if there is an intersection, it is unique. If there happens to be indeed one intersection, then we can define the function τ_i that links t to the parameter of the intersection point in the interval I_i. The domain of definition $\mathbf{\Lambda}_i$ of τ_i is then the set of parameters t that lead to tangents that intersect $\gamma[I_i]$ but no double points :

$$\mathbf{\Lambda}_i := \left\{t \in \mathbb{S}^1 \mid \ |\left\{u \in \mathbb{R}^2 | T_t(u) = 0 \wedge u \in \gamma\left[I_i\right]\right\} \cap \left\{u \in \Gamma \mid \ |\gamma^{-1}[\{u\}]| = 1\right\}| = 1\right\}$$

We then define τ_i as the function that links such a parameter t to the parameter of the only intersection point u in the interval I_i :

$$\tau_i : \begin{cases} \mathbf{\Lambda}_i & \longrightarrow I_i \\ t & \longmapsto \gamma^{-1}(u) \end{cases}$$

Conveniently τ_i is continuous : using the fact that γ is smooth we can proove by contradiction that a discontinuity would lead to another point of intersection.

Differentiating $T_t(\tau_i(t)) = 0$ gives us an expression of its derivative :

$$\begin{aligned}
\tau_i'(t) &= \frac{\left[\gamma\circ\tau_i(t) - \gamma(t), \gamma''(t)\right]}{\left[\gamma'(t), \gamma'\circ\tau_i(t)\right]} \\
&= \frac{\pm\left\|\gamma\circ\tau_i(t) - \gamma(t)\right\| \cdot \left[\gamma'(t), \gamma''(t)\right]}{\left\|\gamma'(t)\right\| \cdot \left[\gamma'(t), \gamma'\circ\tau_i(t)\right]} \\
&= \frac{\pm\left\|\gamma\circ\tau_i(t) - \gamma(t)\right\| \cdot \left\|\gamma'(t)\right\|^3 \cdot \kappa(t)}{\left\|\gamma'(t)\right\|^2 \cdot \left\|\gamma'\circ\tau_i(t)\right\| \cdot \left[\frac{\gamma'(t)}{\|\gamma'(t)\|}, \frac{\gamma'\circ\tau_i(t)}{\|\gamma'\circ\tau_i(t)\|}\right]} \\
&= \pm\frac{\left\|\gamma'(t)\right\|}{\left\|\gamma'\circ\tau_i(t)\right\|} \cdot \frac{\left\|\gamma\circ\tau_i(t) - \gamma(t)\right\| \cdot \kappa(t)}{\left[\frac{\gamma'(t)}{\|\gamma'(t)\|}, \frac{\gamma'\circ\tau_i(t)}{\|\gamma'\circ\tau_i(t)\|}\right]}
\end{aligned}$$

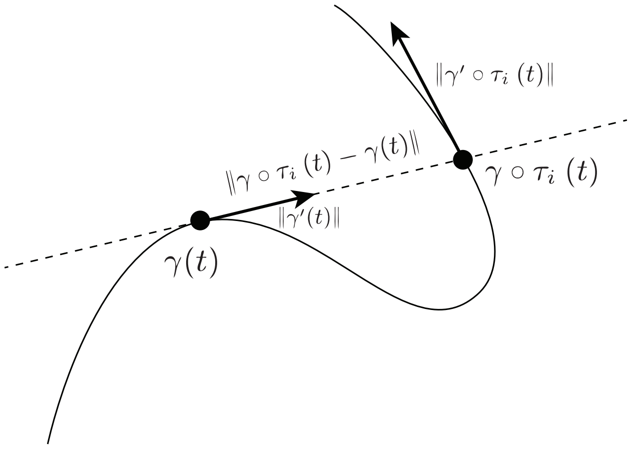

Figure 3: Function τ

Thus τ_i is differentiable everywhere exept at parameters that are part of a bitangent pair (and get sent on the other parameter of the pair by τ_i), in this particular instance the derivative goes to $\pm\infty$.

(Derivative of τ_i).

$$\tau_i'(t) = \pm\frac{\left\|\gamma'(t)\right\|}{\left\|\gamma'\circ\tau_i(t)\right\|} \cdot \frac{\left\|\gamma\circ\tau_i(t) - \gamma(t)\right\| \cdot \kappa(t)}{\left[\frac{\gamma'(t)}{\|\gamma'(t)\|}, \frac{\gamma'\circ\tau_i(t)}{\|\gamma'\circ\tau_i(t)\|}\right]} \tag{1}$$

Remarks :

- $\left[\frac{\gamma'(t)}{\|\gamma'(t)\|}, \frac{\gamma'\circ\tau_i(t)}{\|\gamma'\circ\tau_i(t)\|}\right] = sin\left(\int_t^{\tau_i(t)} \kappa(s).ds\right)$ is simply the sinus of the angle between $\gamma'(t)$ and $\gamma'\circ\tau_i(t)$.
- The sign of the "$\pm$" is that of $\langle\gamma\circ\tau_i(t) - \gamma(t), \gamma'(t)\rangle$, i.e. it's a "+" if we're dealing with a positive tangent ray and a "$-$" if we're dealing with a negative tangent ray.

This expression is very useful in which it provides us with an easy way to discuss the sign of $\tau_i^{n\,\prime}(t)$.

Because we're only concerned with the geometry of the problem we will assume from now on that our arc is parametrised by arc-length and we won't bother with the first term in (1).

If we consider $t \in \mathbf{\Lambda} := \cup\mathbf{\Lambda}_i$ (i.e. all the points for which a τ function can be define), and take a look at all the couples of the form $(t, \tau_k(t))$, $k \in [\![1, 2(p+1)]\!]$ we should be looking at c closed shapes on a torus, where c is the number of convex parts of our curve. Figure 1.2 and 1.3 show an example for one particular plane curve with 2 inflection points and one double point. In this particular case, c=2. The graph is either in full of dashed lines depending on whether the intersection point between the tangent at t and the curve was obtained through a positive or a negative tangent ray.

Given the expression of the derivative for the τ functions we already knew that when we cross a bitangent (and go from a certain τ function to the next) their derivative will jump from $\pm\infty$ to $-\pm\infty$. This can be seen very clearly on figure 1.3. What's more, we can easily identify the type of bitangent (internal or external) with the graph : on the part that corresponds to positive tangent rays the external bitangent are the "outward" angles while the "inwards" angles correspond to internal bitangent, this is reversed when we switch to the part that corresponds to negative tangent rays. The succession of "inward" and "outward" angles is the embodiment of Fabricius-Bjerre's theorem and is closely related to its original proof.

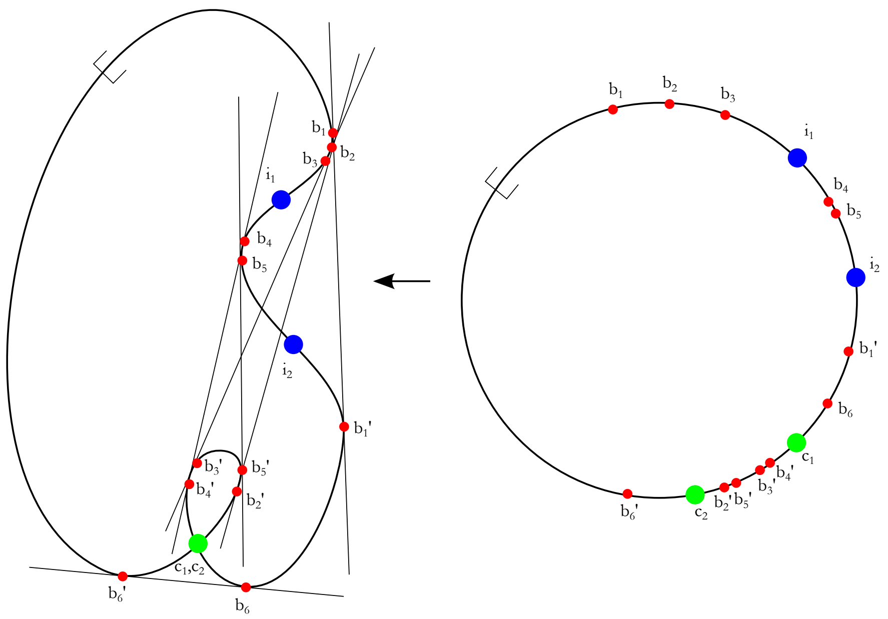

Figure 4: Partition in $2(p+1)$ intervals

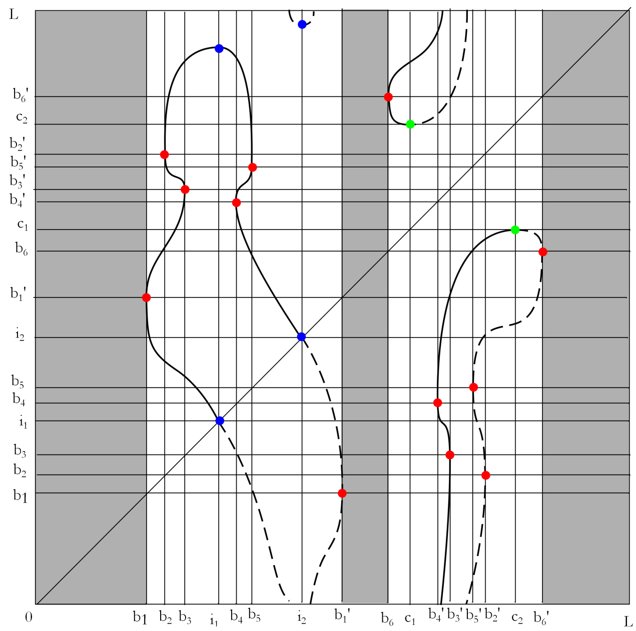

Figure 5: The graph of all the $2(p+1)$ associated τ functions form c disjoint closed graphs

Case study/Inflexion wells

Because it allows us to deal with a single τ function and not to bother about multiple intersections, we look into what happens in the simple configuration where two inflexion points dig out some a well into the curve. We restrict the curvature between the two inflection points to obtain a simpler transition graph of the system and not bother with more cumbersome intervals. This is captured by the following restrictions:

(Inflexion well). We say that the curve γ has an inflexion well if it has a bitangent pair $\{s, t\}$ such that :

(i) the interval $[min\{s,t\}, max\{s,t\}]$ contains exactly two inflexion points and no double points

(ii) $\displaystyle\int_{min\{s,t\}}^{max\{s,t\}} |\kappa(s)| .ds \leqslant 2\pi$

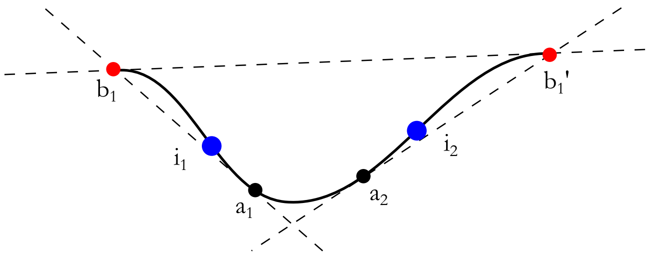

Figure 6: Inflexion well

So we assume that γ has an inflexion well $[b, b']$. Because we're not concerned for the moment about the interaction with the rest of the curve we will assume that this interval contains no other parameter belonging to a bitangent pair. We can then work with a unique τ and resort to all the results known for one-dimensional maps. Using (1) we can sketch what τ should look like (fig. 5). We will refer to $\tau_{|[b,a_1]}$ as the left component of τ and to $\tau_{|[a_2,b']}$ as the right component of τ.

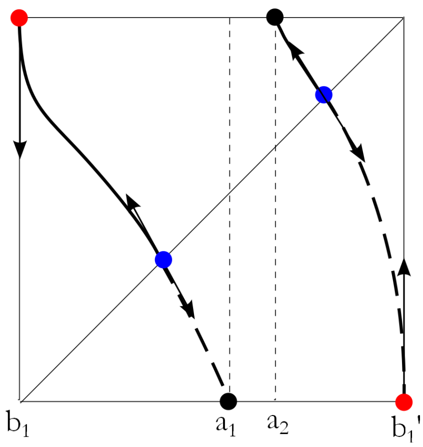

Figure 7: Associated function τ

The study of this system can be reduced to that of its topological linear conjugate. We find that the behavior of τ is essentially that of the tent map save for the fact that its two branches are orientation preserving. Down below are listed a few relevant results that come out from this study :

(The stable set $\mathbf{\Delta}$ is a Cantor set). The stable subset $\mathbf{\Delta}$ for the map τ is the set of points which never escape $\mathbf{\Lambda}$ under forward iteration (which corresponds ultimately to the set of points which never escape the inflexion well). We thus define $\mathbf{\Delta}_k := \bigcap_{k=1}^{\infty} \left\{x \in [b, b'] | \tau^k(x) \in [b, b']\right\}$ for all $k \in \mathbb{N}$ and set $\mathbf{\Delta} := \bigcap_{k=1}^{\infty} \mathbf{\Delta}_k$.
We claim that $\mathbf{\Delta}$ is a Cantor set. As such $\mathbf{\Delta}$ is compact, perfect, nowhere dense and totally disconnected. Thus in an inflexion well there are no intervals whose image by γ is entirely made up of n-gons

Remark. We can easily solve the sum of the intervals subtracted at each iterations for the linear conjugate of τ and we obtain that the stable set is of measure 0 and is therefore a canonical Cantor set.

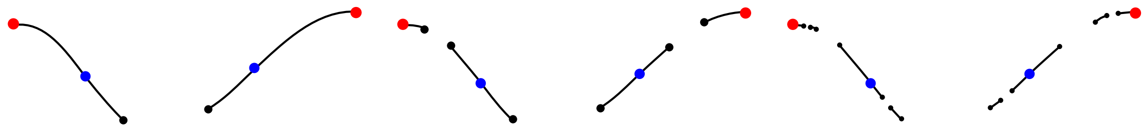

Figure 8: $\gamma[\mathbf{\Delta}_1]$, $\gamma[\mathbf{\Delta}_2]$ and $\gamma[\mathbf{\Delta}_3]$

(The set of periodic points of τ is countable). Define $P_k(\tau := \{x \,|\tau^k = x\})$ for all $k \in \mathbb{N}$ and set $\mathrm{P}(\tau) = \bigcup_{k=1}^{\infty} P_k(\tau)$. Then $\forall k \in \mathbb{N}, \#P_k(\tau) \in \mathbb{N}$ and $\mathrm{P}(\tau)$ is a countable set.

(τ is chaotic on $\mathbf{\Delta}$). All the periodic points of τ are sources and τ has chaotic orbits.

We have the following transition graph with the sub-intervals $I = [b, i_1]$, $J = [i_1, i_2]$ and $K = [i_2, b']$ for the map τ :

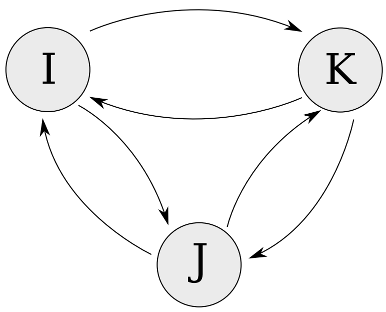

Figure 9: Transition graph for τ

From this we deduce the following result :

(Existence). For every integer n, we are guaranteed the existence of at least one tangent n-gon inscribed in the inflexion well.

Proof. For odd integers n select the path $I(KJ)^{\lfloor \frac{n}{2} \rfloor}I$.
For odd integers n select the path $IK(JK)^{\lfloor \frac{n}{2} \rfloor}I$.
By construction these paths cannot correspond to periodic points of lower period.

References

[1] Barbara Jablonska. Surface associated to a space curve : A new proof of Fabricius-Bjerre Formula.
https://depositonce.tu-berlin.de/handle/11303/3520

[2] Thomas F. Banchoff. Global geometry of polygons. I: The theorem of Fabricius-Bjerre.
http://www.ams.org/journals/proc/1974-045-02/S0002-9939-1974-0370599-7/S0002-9939-1974-0370599-7.pdf

[3] Benjamin Halpern. Global theorems for closed plane curves.
http://www.ams.org/journals/bull/1970-76-01/S0002-9904-1970-12380-1/S0002-9904-1970-12380-1.pdf

[4] Fr. Fabricius-Bjerre. On the double tangents of plane closed curves.
http://www.ams.org/journals/bull/1970-76-01/S0002-9904-1970-12380-1/S0002-9904-1970-12380-1.pdf

[5] Marcel Berger & Bernard Gostiaux. Differential Geometry: Manifolds, Curves, and Surfaces.
http://www.springer.com/us/book/9780387966267

[6] Marcel Berger. Geometry Revealed: A Jacob's Ladder to Modern Higher Geometry.
http://www.springer.com/us/book/9783540709961

[7] Geoffrey R. Goodson. Chaotic Dynamics: Fractals, Tilings and Substitutions.
http://pages.towson.edu/goodson/Chaos_Notes.pdf

[8] Steven H. Strogatz. Non Linear Dynamics and Chaos.
http://planets.utsc.utoronto.ca/~pawel/PHYD38/Strogatz.pdf

[9] Kathleen T. Alligood, Tim D. Sauer, James A. Yorke. CHAOS : An Introduction to Dynamical Systems.
http://www.springer.com/la/book/9780387946771

[10] Roger L. Kraft. Chaos, Cantor Sets, and Hyperbolicity for the Logistic Maps.
http://www.jstor.org/stable/2589144?seq=1#page_scan_tab_contents

[11] Aliasghar Alikhani-Koopaei. On the Set of Fixed Points and Periodic Points of Continuously Differentiable Functions.
http://www.hindawi.com/journals/ijmms/2013/929475/#B1